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icosahedron rotation group and it is the smallest simple group of composite order. The fifth is very well known in the theory of elliptic modular functions and is the second smallest simple group of composite order. Kirkman remarks:* "Betti, Kronecker, Hermite, and myself have spent much time on this group." The last of these six primitive groups is the holomorph of the group of order 8 which includes no operator of order four.†

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*Kirkman, Proceedings of the Manchester Literary and Philosophical Society, vol. 3, p. 65.
†American Journal of Mathematics, loc. cit.

Leland Stanford University.

FACTORS OF A CERTAIN DETERMINANT OF OREER SIX.

By DR. L. E. DICKSON.

The following is an example of the so-called Group-Determinant:*

$$D\equiv\begin{vmatrix} \mathrm{I} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \beta & \mathrm{I} & \alpha & \delta & \varepsilon & \gamma \\ \alpha & \beta & \mathrm{I} & \varepsilon & \gamma & \delta \\ \gamma & \delta & \varepsilon & \mathrm{I} & \alpha & \beta \\ \delta & \varepsilon & \gamma & \beta & \mathrm{I} & \alpha \\ \varepsilon & \gamma & \delta & \alpha & \beta & \mathrm{I} \end{vmatrix}$$

Note that the elements of the first three rows form two cyclic determinants of order three, and that the elements of the last three rows form the same two cyclic determinants. It follows readily that D has the factors $(\mathrm{I}+\alpha+\beta)\pm(\gamma+\delta+\varepsilon)$.

Upon adding to the first column all the remaining columns, we obtain an equal determinant having $\mathrm{I}+\alpha+\beta+\gamma+\delta+\varepsilon$ throughout the first column. Let D_1 be the determinant obtained by removing this factor, so that the elements in the first column of D_1 are all unity. Subtracting the first row from the remaining rows, we find that

$$D_1=\begin{vmatrix} \mathrm{I}-\alpha & \alpha-\beta & \delta-\gamma & \varepsilon-\delta & \gamma-\varepsilon \\ \beta-\alpha & \mathrm{I}-\beta & \varepsilon-\gamma & \gamma-\delta & \delta-\varepsilon \\ \delta-\alpha & \varepsilon-\beta & \mathrm{I}-\gamma & \alpha-\delta & \beta-\varepsilon \\ \varepsilon-\alpha & \gamma-\beta & \beta-\gamma & \mathrm{I}-\delta & \alpha-\varepsilon \\ \gamma-\alpha & \delta-\beta & \alpha-\gamma & \beta-\delta & \mathrm{I}-\varepsilon \end{vmatrix}$$

From the first row, subtract the third, fourth, and fifth rows, and to the first row add the second row. In the resulting determinant, the elements of the first row are all divisible by $\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon$. Hence

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*Its matrix forms the body of a left-hand multiplication-table for the symmetric group on three letters, where

$$\mathrm{I}=\text{identity},\ \alpha=(123),\ \beta=(132),\ \gamma=(12),\ \delta=(13),\ \varepsilon=(23).$$

Compare Weber, *Algebra*, 2nd Edition, Vol, II, page 124.

$$D_1=(\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon)D_2,$$

where the elements in the first row of D_2 are 1, 1, -1, -1, -1; the remaining rows of D_2 being identical with the corresponding rows of D_1. Next subtract the first column of D_2 from the second column, and add the first column to the third, fourth, and fifth columns.

$$\therefore\ D_2=\begin{vmatrix} \mathrm{I}+\alpha-2\beta & -\alpha+\beta-\gamma+\varepsilon & -\alpha+\beta+\gamma-\delta & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta-\delta+\varepsilon & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta+\gamma-\varepsilon & -\alpha+\beta-\gamma+\varepsilon & \mathrm{I}-\alpha-\delta+\varepsilon & 0 \\ \alpha-\beta-\gamma+\delta & 0 & -\alpha+\beta+\gamma-\delta & \mathrm{I}-\alpha+\gamma-\varepsilon \end{vmatrix}$$

Upon subtracting the second row from the first, the new first row is

$$\mathrm{I}-\beta+\delta-\varepsilon \quad -\mathrm{I}+\beta-\delta+\varepsilon \quad -\alpha+\beta+\gamma-\delta \quad 0$$

Add the fourth column to the first and the new first column to the second.

$$\therefore\ D_2=\begin{vmatrix} \mathrm{I}-\beta+\delta-\varepsilon & 0 & -\alpha+\beta+\gamma-\delta & 0 \\ 0 & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta+\delta-\varepsilon & 0 & \mathrm{I}-\alpha-\delta+\varepsilon & 0 \\ \mathrm{I}-\beta+\delta-\epsilon & \mathrm{I}-\beta+\delta-\epsilon & -\alpha+\beta+\gamma-\delta & \mathrm{I}-\alpha+\gamma-\varepsilon \end{vmatrix}$$

Subtract the first row from the third and fourth rows. In the resulting determinant, subtract the second column from the fourth column. Hence

$$D_2=\begin{vmatrix} \mathrm{I}-\beta+\delta-\epsilon & 0 & -\alpha+\beta+\gamma-\delta & 0 \\ 0 & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\mathrm{I}+\beta+\gamma-\varepsilon \\ -\mathrm{I}+\alpha+\gamma-\delta & 0 & \mathrm{I}-\beta-\gamma+\epsilon & 0 \\ 0 & \mathrm{I}-\beta+\delta-\epsilon & 0 & -\alpha+\beta+\gamma-\delta \end{vmatrix}$$

By inspection, it is seen that this determinant is the square of

$$\triangle\equiv\begin{vmatrix} \mathrm{I}-\beta+\delta-\epsilon & -\alpha+\beta+\gamma-\delta \\ -\mathrm{I}+\alpha+\gamma-\delta & \mathrm{I}-\beta-\gamma+\varepsilon \end{vmatrix}$$

$$=\mathrm{I}^2+\alpha^2+\beta^2-\mathrm{I}\alpha-\mathrm{I}\beta-\alpha\beta-\gamma^2-\delta^2-\epsilon^2+\gamma\delta+\gamma\varepsilon+\delta\epsilon.$$

To show that $\triangle$ is algebraically irreducible, we note that

$$\triangle=(\mathrm{I}+\omega\alpha+\omega^2\beta)(\mathrm{I}+\omega^2\alpha+\omega\beta)-(\gamma+\omega\delta+\omega^2\epsilon)(\gamma+\omega^2\delta+\omega\varepsilon),$$

where ω is an imaginary cube root of unity. But the expressions

$$x=\mathrm{I}+\omega\alpha+\omega^2\beta,\quad y=\mathrm{I}+\omega^2\alpha+\omega\beta,\quad z=\gamma+\omega\delta+\omega^2\epsilon,\quad w=\gamma+\omega^2\delta+\omega\varepsilon$$

are independent functions of I, α, β, γ, δ, ϵ. Hence $\triangle\equiv xy-zw$ is irreducible.

Hence the decomposition of the given determinant of order six into algebraically irreducible factors is as follows: *

$$D=(1+\alpha+\beta+\gamma+\delta+\varepsilon)(1+\alpha+\beta-\gamma-\delta-\varepsilon)\triangle^2.$$

In particular, D is expressible as the difference of two squares,

$$D=[(1+\alpha+\beta)\triangle]^2-[(\gamma+\delta+\epsilon)\triangle]^2.$$

The fact that $\triangle$ is a factor of D may be shown directly. Multiply the second row of D by ω and the third row by ω^2 and add the products to the first row. Multiply the fifth row by ω^2 and the sixth row by ω and add the products to the fourth row. The new first and fourth rows are

$$\begin{array}{cccccc} y & \omega y & \omega^2 y & z & \omega^2 z & \omega z, \\ w & \omega w & \omega^2 w & x & \omega^2 x & \omega x. \end{array}$$

Then develop the determinant by Laplace's method. Each term is the product of a determinant of order two formed from the above two rows by the complementary determinant of order four formed from the remaining rows. Each of these determinants of order two has the factor $xy-zw\equiv\triangle$. Hence D has the factor $\triangle$.

The University of Chicago, January, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

153. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Find some two-figure numbers, such that if they be squared, then the figures interchanged and the resulting numbers squared, the resulting products will consist of the same digits in reversed order.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The difference of the two numbers is a multiple of 9.

$\therefore 10x+y=10y+x+9n.$

$\therefore x=y+n.$

Let $n=0$. The numbers are 11, 11; 22, 22.

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*Since writing this paper, I find that the result was obtained in 1886 by Dedekind by the theory of matrices (Berliner Sitzungsberichte, 1897, page 1007).